Using Symbolic Computation in the Description Problem for Degenerate Homogeneous Hypersurfaces in $\mathbb{C}^4$

Loboda A.V., Sukovykh V.I. (Voronezh, Russia)

* This work was supported by the Russian Foundation for Basic Research (project No 20-01-00497-a) and the Moscow Center for Fundamental and Applied Mathematics, Lomonosov Moscow State University

Dubna, 24-25 May 2021
1. History

The presented work is related to the study of holomorphically homogeneous real hypersurfaces of a 4-dimensional complex space.

1. The cases of $\mathbb{C}^2$ and $\mathbb{C}^3$ are completely studied:


2. In $\mathbb{C}^4$ there are some works related both to Levi non-degenerate and degenerate cases.


2. Lie Algebras

Each such surface has at least 7-dimensional Lie algebra of holomorphic vector fields.

Unfortunately, there are no complete lists of (abstract) 7-dimensional Lie algebras. In 2D and 3D cases, such lists of 3-dimensional and 5-dimensional Lie algebras exist. Both the work of E. Cartan, and the classification of homogeneous surfaces in $\mathbb{C}^3$ use these lists.

It is time to construct such lists for 7-dimensional Lie algebras. This task can only be solved using computer algorithms.

But nowadays, there are only partial lists (for example, the description of 7-dimensional nilpotent Lie algebras [Gong M.P.]) and some obvious families of homogeneous hypersurfaces.

**Examples.**

1. $\Gamma^5 \times \mathbb{C}$,
2. $M^3 + i\mathbb{R}^4$.

**Remark.** There is no complete description of affinely homogeneous $M^3 \in \mathbb{R}^4$. 
3. Tubes with degenerate bases and vector fields

Let M be analytic hypersurface in \( \mathbb{R}^4 \):

\[
M = \{ x_4 = F(x_1, x_2, x_3), \quad dF(0) = 0 \} \subset \mathbb{R}^4
\]  

(1.1)

or

\[
x_4 = F_2(x_1, x_2, x_3) + F_3(x_1, x_2, x_3) + F_4(x_1, x_2, x_3) + \ldots,
\]

(1.2)

where \( F_2 = x_1^2 + x_2^2 \), \( F_k = F_k(x_1, x_2, x_3) \) for \( k \geq 3 \).

**Remark.** If all the \( F_k \) does not depend on \( x_3 \) we have one of the known homogeneous cylinders \( (\gamma + \mathbb{R}) \).

We assume that the algebra of affine vector fields tangent to M has dimension 3 and the basis fields

\[
E_1 = (1 + L_{11}(x)) \frac{\partial}{\partial x_1} + L_{12}(x) \frac{\partial}{\partial x_2} + L_{13}(x) \frac{\partial}{\partial x_3} + L_{14}(x) \frac{\partial}{\partial x_4},
\]

\[
E_2 = L_{21}(x) \frac{\partial}{\partial x_1} + (1 + L_{22}(x)) \frac{\partial}{\partial x_2} + L_{23}(x) \frac{\partial}{\partial x_3} + L_{24}(x) \frac{\partial}{\partial x_4},
\]

\[
E_3 = L_{31}(x) \frac{\partial}{\partial x_1} + L_{32}(x) \frac{\partial}{\partial x_2} + (1 + L_{33}(x)) \frac{\partial}{\partial x_3} + L_{34}(x) \frac{\partial}{\partial x_4},
\]

(1)

where \( x = (x_1, x_2, x_3, x_4) \),

\( L_{ik}(x) \) \( (1 \leq i \leq 3, \ 1 \leq k \leq 4) \) – linear forms in four variables.

**Tangency condition:**

\[
(E_k(\Phi))|_{\Phi=0} = 0 \quad (k = 1, 2, 3).
\]
4. Algebras of vector fields

\[ E_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 1 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} & 0 \\ b_{21} & b_{22} & b_{23} & b_{24} & 1 \\ b_{31} & b_{32} & b_{33} & b_{34} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ E_3 = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 \\ c_{21} & c_{22} & c_{23} & c_{24} & 0 \\ c_{31} & c_{32} & c_{33} & c_{34} & 1 \\ c_{41} & c_{42} & c_{43} & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]

\[ [E_i, E_j] = E_i E_j - E_j E_i = \alpha E_1 + \beta E_2 + \gamma E_3. \]

\[ L_{1k} = \sum_{j=1}^{4} a_{kj} x_j, \quad L_{2k} = \sum_{j=1}^{4} b_{kj} x_j, \quad L_{3k} = \sum_{j=1}^{4} c_{kj} x_j. \]
5. Affine Normal Form of the Hypersurface

Using some simplification of the surface equation we get a set of restriction to the vector fields. Let

\[ M = \{ x_4 = (x_1^2 + x_2^2) + F_3(x_1, x_2, x_3) + F_4(x_1, x_2, x_3) + \ldots \}, \quad (1.2) \]

and

\[ F_3(x_1, x_2, x_3) = (A_{300}x_1^3 + A_{210}x_1^2x_2 + A_{120}x_1x_2^2 + A_{030}x_2^3) + \]
\[ + (A_{201}x_1^2 + A_{111}x_1x_2 + A_{021}x_2^2)x_3 + (A_{102}x_1 + A_{012}x_2)x_3^2 + A_{003}x_3^3. \quad (1.5) \]

1) After change of variables

\[ x_1^* = x_1 + Ax_4, \quad x_2^* = x_2 + Bx_4 \quad (1.6) \]

and rotation in \((x_1, x_2)\)-plane we get

\[ F_3^{(1)}(x_1, x_2) = \varepsilon(x_2^3 - 3x_1^2x_2). \quad (1.7) \]

2) Term \( F_3 \) must have the form

\[ F_3^{(1)}(x_1, x_2) = a_1(x_2^3 - 3x_1^2x_2) + (a_2x_1^2 + a_3x_1x_2 + a_4x_2^2)x_3. \]
6. Simplification of the basic vector fields

One can consider homogeneous components of analytic equations

\[ (E_k(\Phi))_{\Phi=0} = 0 \quad (k = 1, 2, 3). \]

\[ \text{deg} = 1 : \frac{\partial F_2}{\partial x_1} - l_{14}(x_1, x_2, x_3) = 0. \]

\[ \text{deg} = 2 : \frac{\partial F_3}{\partial x_1} + 2l_{11}(x)x_1 + 2l_{12}(x)x_2 - a_{44}(x_1^2 + x_2^2) = 0. \]

**Corollary.**

\[
E_1 = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & 1 \\
a_{21} & a_{22} & a_{23} & a_{24} & 0 \\
a_{31} & a_{32} & a_{33} & a_{34} & 0 \\
2 & 0 & 0 & & a_{44} \\
\end{pmatrix},
\]

\[
E_2 = \begin{pmatrix}
b_{11} & b_{12} & b_{13} & b_{14} & 0 \\
b_{21} & b_{22} & b_{23} & b_{24} & 1 \\
b_{31} & b_{32} & b_{33} & b_{34} & 0 \\
0 & 2 & 0 & b_{44} & 0 \\
\end{pmatrix}, \quad E_3 = \begin{pmatrix}
c_{11} & c_{12} & 0 & c_{14} & 0 \\
c_{21} & c_{22} & 0 & c_{24} & 0 \\
c_{31} & c_{32} & c_{33} & c_{34} & 1 \\
0 & 0 & 0 & c_{44} & 0 \\
\end{pmatrix}.
\]

Moreover, the 13 linear restrictions on the elements of \( E_1, E_2, E_3 \) and four coefficients of \( F_3 \) hold:

\[ 2a_{11} - a_{44} = 0, \quad 2a_{21} + 2a_{12} - 6a_1 = 0, \quad 2a_{13} + 2a_2 = 0, \quad a_3 + 2a_{23} = 0, \ldots \]

(18)
7. System of quadratic equations

Now we have 3 basis matrix

\[
E_1 = \begin{bmatrix}
    a_{11} & a_{12} & -a_2 & a_{14} & 1 \\
    -a_{12} + 3a_1 & a_{11} & a_{23} & a_{24} & 0 \\
    a_{31} & a_{32} & a_{33} & a_{34} & 0 \\
    2 & 0 & 0 & 2a_{11} & 0 \\
    0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
E_2 = \begin{bmatrix}
    3a_1 + b_{22} & b_{12} & -1/2a_3 & b_{14} & 0 \\
    -b_{12} & b_{22} & -a_4 & b_{24} & 1 \\
    b_{31} & b_{32} & b_{33} & b_{34} & 0 \\
    0 & 2 & 0 & 3a_1 + 2b_{22} & 0 \\
    0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
E_3 = \begin{bmatrix}
    -1/2a_2 + 1/2a_4 + c_{22} & c_{12} & 0 & c_{14} & 0 \\
    -1/2a_3 - c_{12} & c_{22} & 0 & c_{24} & 0 \\
    c_{31} & c_{32} & c_{33} & c_{34} & 1 \\
    0 & 0 & 0 & a_4 + 2c_{22} & 0 \\
    0 & 0 & 0 & 0 & 0
\end{bmatrix}. \quad (19)
\]

Coefficients of expansions

\[
[E_i, E_j] = E_iE_j - E_jE_i = \alpha E_1 + \beta E_2 + \gamma E_3.
\]

can be determined from 5-th columns of commutators. And after that we obtain a system of 48 = 16 × 3 quadratic equations in (24 + 4) = 28 variables. The question arises: how can one find all its solutions?
8. Attempts to solve the System

9 equations of the System are in reality zero identities. So we have 39 equations in 28 variables. The most cumbersome equation of the system contains 14 (quadratic) terms; the simplest ones are here:

\[-a_{12}a_4 - a_2b_{33} + a_3b_{12} + 1/2 a_3a_{33} + a_2a_{12} = 0,\]
\[-a_2c_{33} - a_2^2 + a_3c_{12} = 0,\]
\[-a_4c_{33} - 1/2 a_3^2 - a_3c_{12} - a_4^2 = 0.\]

Groebner bases do not give a solution by using usual notebook (plex, tdeg) in general situation.

Moreover, setting \(a_2 = a_3 = a_4 = 0\) also do not give a solution.

Three years ago, we had an unsuccessful experience in solving systems of polynomial equations of the (6x6x6)-type (number of equations, number of unknowns, degree of equations), although Professor Gerdt believed that such systems could be solved.

Now, in the situation (39,28,2), we also ask for help from the «computer algebra» community.
9. Partial solutions

In some partial cases we get many families of Lie algebras. Some of them depend on 3 real parameters and basis matrixes are tremendous ones; there are more simple families.

Example 1.

\[
E_{1,2,3} : \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 2 & 0 \\
2 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The orbits in \( \mathbb{R}^4 \) of this algebra do not depend on the variable \( x_3 \), so the tubes over them are involved in «trivial» family \( \{ \Gamma^5 \times \mathbb{C} \} \).

Example 2.

\[
\begin{pmatrix}
0 & 0 & -1 & a_{14} & 1 \\
0 & 0 & 0 & 0 & 0 \\
2a_{14} & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
b_{11} & 0 & 0 & 0 & 0 \\
0 & 1 + b_{11} & 0 & 0 & 1 \\
0 & 2a_{14} & 0 & 2b_{11}a_{14} & 0 \\
0 & 2 & 0 & 2b_{11} & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
-1/2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & a_{14} & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
10. New homogeneous surfaces

To obtain the orbits of the constructed Lie algebras, they need to be integrated. For the multi-parameter families of algebras, this is a rather long procedure (it is also convenient to do this by means of symbolic computations). Below is a one family of such orbits in the space $\mathbb{R}^4$:

$$x_4x_3 + x_1^2 = x_3(1 + x_2)^\alpha, \ \alpha \in \mathbb{R} \setminus \{0, 1\},$$

$$x_4x_3 + x_1^2 = x_3 \ln(1 + x_2),$$

$$x_4x_3 + x_1^2 = x_3 \exp(x_2).$$

The tubes over them are (new) degenerate homogeneous hypersurfaces in $\mathbb{C}^4$. They all depend on the complete set of variables in $\mathbb{C}^4$, i.e. are nontrivial (do not belong to the family $\{\Gamma^5 \times \mathbb{C}\}$).

**Remark.** These surfaces correspond to the case

$$a_1 = 1, \ a_2 = a_3 = 0, \ a_4 = -3/2$$

and were found «almost accidentally».

The search for systematic approach to the problem continues ... .

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Some additional references


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THANK FOR YOUR ATTENTION!