An explicit procedure for calculating the perimeter of an ellipse

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Statement of purpose

We discuss a concise, quadratically convergent, algorithm for calculating the perimeter of an ellipse, based on the concept of the generalized arithmetic-geometric mean (GAGM) which was shown to yield the modified arithmetic-geometric mean (MAGM) and the arithmetic-geometric mean (AGM) as two special cases.

We aim to incorporate this algorithm into the computer algebra system (CAS) MathPartner, which was founded by Gennadi Malaschonok.

The formula for calculating the perimeter of an ellipse

Assume, unless indicated otherwise, that β and γ are two positive numbers which squares sum to one: $\beta^2 + \gamma^2 = 1$.

Recall that the complete elliptic integral of the second kind might be expressed as

$$\int_0^1 \sqrt{\frac{1-\gamma^2 x^2}{1-x^2}}\, dx = \frac{\pi N(\beta^2)}{2\,M(\beta)}, \tag{1}$$

where $M(x)$ is the arithmetic-geometric mean of x and 1, whereas $N(x)$ is the modified arithmetic-geometric mean of x and 1.

The formula given in (1) was shown to enable the calculation of the perimeter of an ellipse via a quadratically convergent iterative procedure.

An iterative procedure for simultaneous calculation of the AGM and MAGM

The algorithm which we aim to discuss enables a "simultaneous" calculation of the AGM and the MAGM via a "unifying" iterative procedure, requiring at each step (aside from basic arithmetic operations) a single square root extraction. In other words, we outline an iterative algorithm for calculating the ratio $N(\beta^2)/M(\beta)$, which upon multiplying it by $\pi/2$ would yield the corresponding complete elliptic integral of the second kind, according to formula (1).

Our ability to iteratively calculate the said ratio $N(\beta^2)/M(\beta)$ is based on the observation that the AGM, along with the MAGM is, indeed, a special case of the GAGM. In fact, we have

$$M(\beta) = N\left(\beta,\, \frac{1}{\beta},\, 0,\, \beta - i\,\gamma,\, 1,\, \beta + i\,\gamma\right),$$ [1]

that is, the AGM of β and 1 would coincide with the GAGM of β and $1/\beta$ for parameters $\beta - i\,\gamma$, 1 and $\beta + i\,\gamma$.

[1] We have noted that if β, $1/\beta$ and 0 are regarded to be the values of the three half-period of an essential elliptic function, then $\beta - i\,\gamma$, 1 and $\beta + i\,\gamma$ are the values at three (out of six) quarter-periods of the same (Galois) function.

An algorithm for calculating the perimeter of an ellipse

Put

$$\{x_0,\, y_0,\, z_0\} = \{\beta,\, \frac{1}{\beta},\, 0\},\; r_0 = 1,\; \rho_1 = 1,$$ [2]

and recursively calculate

$$\{x_{n+1},\, y_{n+1},\, z_{n+1}\} = \left\{\frac{x_n + y_n}{2},\, z_n + r_n,\, z_n - r_n\right\},\; \rho_{n+1} = \rho_n\,\frac{x_{n-1} - z_n}{x_n - z_n},$$

where $r_n = \sqrt{2(x_n - z_n)\,r_{n-1}}$, then (for each positive integer n) the interval

$$\rho_n\,(x_n,\, x_{n-1}) \tag{2}$$

must contain the ratio $N(\beta^2)/M(\beta)$.

[2] Instead, we could have started with $x_{-1} = \beta - i\,\gamma$, $r_{-1} = 1/(2\beta)$, and $\rho_0 = \beta(\beta + i\,\gamma)$.

The perimeter of an ellipse, expressed as a product of two concomitant limits

Note, as well, that the length is of the interval (2) is a product of two multiplicands: ρ_n and $(x_{n-1} - y_{n-1})/2$, the first of which converges to $\beta/M(\beta)$, whereas the second decreases (quadratically) to zero, as the (descending) sequence $\{x_n\}$ and the (ascending) sequence $\{y_n\}$ approach their (common) limit $N(\beta^2)/\beta$.

So, denoting by ρ and x the respective limits of the sequences $\{\rho_n\}$ and $\{x_n\}$, we have

$$\int_0^1 \sqrt{\frac{1-\gamma^2 x^2}{1-x^2}}\, dx = \frac{\pi\,\rho\,x}{2}.$$

An exemplary algorithm for computing π

Adopting the formula

$$\pi = \frac{M^2}{N-1},$$

where $M = M(\sqrt{2})$ and $N = N(2)$, we set

$$\{x_0,\, y_0,\, z_0\} = \{2,\, 1,\, 0\},\ r_0 = \sqrt{2},\ \rho_1 = 1/\sqrt{2}$$

and apply the preceding recursive formulas in order to successively calculate the intervals

$$\pi_n = \frac{1}{\rho_n^2}\left(\frac{1}{x_{n-1}-1}, \frac{1}{x_n-1}\right),$$

containing the constant π, where $1/\rho_n$ converges to M: $\pi_1 = (2, 4)$,

$$\pi_2 \approx (2.914213562373095049,\ 3.187672642712108627),$$

$$\pi_3 \approx (3.140579250522168248,\ 3.141680293297653294),$$

$$\pi_4 \approx (3.141592646213542282,\ 3.141592653895446496),$$

$$\pi_5 \approx (3.141592653589793238,\ 3.141592653589793238).$$

A clarification

This work aims to dispel a widespread misapprehension concerning the calculation of elliptic integrals. Richard Brent, while (futilely) disputing the priority of the Gauss-Euler algorithm (for calculating π) and emphasizing "modern algorithms", "forgets" to credit Anatoly Karatsuba for his pioneering contribution to fast multiplication techniques. We still are regularly "reminded" that "no simple exact closed formula for calculating the perimeter of an ellipse exists". Some "popular" sources go on even further telling us that "no exact formula is possible"! Matt Parker reiterates the belief that "there is no neat equation for the perimeter of an ellipse" and that "there is no well-defined equation". We hope that presenting a concrete simultaneously simple and quadratically convergent algorithm may facilitate an understanding of the availability of supreme techniques for calculating elliptic integrals. So, unlike any (linearly convergent) approximation, the presented (exact and robust) algorithm does not require any additional "error estimate".

Several indefinite integrals of elliptic functions (source I)

Digital Library of Mathematical Functions

§22.14(iii) Other Indefinite Integrals

In (22.14.13)–(22.14.15), $0 < x < 2K$.

22.14.13 $$\int \frac{dx}{\operatorname{sn}(x,k)} = \ln\left(\frac{\operatorname{sn}(x,k)}{\operatorname{cn}(x,k)+\operatorname{dn}(x,k)}\right),$$

22.14.14 $$\int \frac{\operatorname{cn}(x,k)\,dx}{\operatorname{sn}(x,k)} = \frac{1}{2}\ln\left(\frac{1-\operatorname{dn}(x,k)}{1+\operatorname{dn}(x,k)}\right),$$

22.14.15 $$\int \frac{\operatorname{cn}(x,k)\,dx}{\operatorname{sn}^2(x,k)} = -\frac{\operatorname{dn}(x,k)}{\operatorname{sn}(x,k)}.$$

Several indefinite integrals of elliptic functions (source II)

- $$\int \frac{1}{\operatorname{sn}(z\,|\,m)}\,dz = \log\left(\frac{\operatorname{sn}(z\,|\,m)}{\operatorname{cn}(z\,|\,m)+\operatorname{dn}(z\,|\,m)}\right)$$

- $$\int \frac{dz}{\operatorname{sn}(z\,|\,m)^2} = z - \frac{\operatorname{cn}(z\,|\,m)\operatorname{dn}(z\,|\,m)}{\operatorname{sn}(z\,|\,m)} + \frac{\left(m\operatorname{sn}(z\,|\,m)^2-1\right)E(\operatorname{am}(z\,|\,m)\,|\,m)}{\operatorname{dn}(z\,|\,m)\sqrt{1-m\operatorname{sn}(z\,|\,m)^2}}$$

- $$\int \frac{1}{\operatorname{sn}(z\,|\,m)^3}\,dz = \frac{1}{2}\left((m+1)\log\left(\frac{\operatorname{sn}(z\,|\,m)}{\operatorname{cn}(z\,|\,m)+\operatorname{dn}(z\,|\,m)}\right) - \frac{\operatorname{cn}(z\,|\,m)\operatorname{dn}(z\,|\,m)}{\operatorname{sn}(z\,|\,m)^2}\right)$$

Several definite (modulo $\pi\sqrt{-1}$) integrals of elliptic functions (version I)

The pair of definite integrals

$$\int_{\frac{\pi}{2M\left(\sqrt{\beta},\sqrt{\beta-1/\beta}\right)}}^{x} \frac{dx}{\mathcal{S}(x,\beta)} = \ln \mathcal{S}\left(\frac{\sqrt{\beta-1/\beta}\,x}{2}, \frac{1-\beta}{1+\beta}\right),$$

$$\int_{\frac{\pi\sqrt{\beta}}{2M\left(\sqrt{1-\beta^2}\right)}}^{x} \mathcal{S}(x,\beta)\,dx = \ln \mathcal{S}\left(\frac{\sqrt{\beta-1/\beta}}{2}\left(x+\frac{\pi\sqrt{-\beta}}{2M(\beta)}\right), \frac{1-\beta}{1+\beta}\right),$$

where $M(x,y)$ is the arithmetic-geometric mean of x and y and $M(x) := M(1,x)$, might be rewritten as

$$\int_{\frac{\pi}{2M(\beta-1,\beta+1)}}^{x} \frac{dx}{\mathrm{sn}(x,\beta)} = \ln \mathcal{S}\left(\frac{\sqrt{\beta^2-1}\,x}{2}, \frac{1-\beta}{1+\beta}\right),$$

$$\int_{\frac{\pi}{2M\left(\sqrt{1-\beta^2}\right)}}^{x} \beta\,\mathrm{sn}(x,\beta)\,dx = \ln \mathcal{S}\left(\frac{\sqrt{\beta^2-1}}{2}\left(x+\frac{\pi\sqrt{-1}}{2M(\beta)}\right), \frac{1-\beta}{1+\beta}\right).$$

Alternatively, the preceding integrals might be expressed as

$$\int_{\frac{\pi}{2\sqrt{-1}M(\beta)}}^{x} \frac{dx}{\mathrm{sn}(x,\beta)} = \ln \mathrm{sn}\left(\frac{\sqrt{-1}\,(1+\beta)\,x}{2}, \frac{1-\beta}{1+\beta}\right),$$

$$\int_{\frac{\pi}{2\sqrt{-1}M(-\beta)}}^{x} \frac{dx}{\mathrm{sn}(x,\beta)} = \ln \mathrm{sn}\left(\frac{\sqrt{-1}\,(1-\beta)\,x}{2}, \frac{1+\beta}{1-\beta}\right),$$

$$\int_0^x \beta\,\mathrm{sn}(x,\beta)\; dx = \ln \mathrm{sn}\left(\frac{1+\beta}{2}\left(\sqrt{-1}\,x + \frac{\pi}{2M(\beta)}\right), \frac{1-\beta}{1+\beta}\right) =$$

$$= -\ln \mathrm{sn}\left(\frac{1-\beta}{2}\left(\sqrt{-1}\,x + \frac{\pi}{2M(-\beta)}\right), \frac{1+\beta}{1-\beta}\right).$$

Several integrals of "degenerate" elliptic functions

An improper special case, corresponding to $\beta = 0$, would be

$$\int_{\infty}^{x} \frac{dx}{\sin(x)} = \ln \tanh\left(\frac{\sqrt{-1}\,x}{2}\right) = \ln \tan\left(\frac{x}{2}\right) + \ln\sqrt{-1}.$$

Definite special cases, corresponding to $\beta = \pm 1$, are

$$\int_{\frac{\pi}{2\sqrt{-1}}}^{x} \frac{dx}{\tanh(x)} = \ln \sin\left(\sqrt{-1}\,x\right) = \ln \sinh(x) + \ln\sqrt{-1},$$

$$\int_{\frac{\pi}{2}}^{x} \frac{dx}{\tan(x)} = \ln \sin(x)\,,$$

$$\int_{0}^{x} \tanh(x)\,dx = \ln \sin\left(\sqrt{-1}\,x + \frac{\pi}{2}\right) = \ln \cosh(x),$$

$$-\int_{0}^{x} \tan(x)\,dx = \ln \sin\left(x + \frac{\pi}{2}\right) = \ln \cos(x).$$

One “last” integral of a “degenerate” elliptic function

We must reveal that the (square) values

$$\mathrm{sn}\left(\frac{1+\beta}{2}\left(\sqrt{-1}\,x+\frac{\pi}{2M(\beta)}\right),\frac{1-\beta}{1+\beta}\right)^2,\ \mathrm{sn}\left(\frac{1+\beta}{2}\sqrt{-1}\,x,\frac{1-\beta}{1+\beta}\right)^2$$

are interrelated via the inversion

$$x \mapsto t_\beta(x) := \frac{x-1}{x(1-\beta)^2/(1+\beta)^2-1}$$

before we consider “a last” case of “degeneration” for which β is (again) tending to zero, so

$$\int_0^x \beta\sin(x)\,dx \approx \frac{1}{2}\ln t_\beta\left(\tanh\left(\frac{\sqrt{-1}\,x}{2}\right)^2\right) = \frac{1}{2}\ln\left(\frac{(1+\beta)^2}{1+\beta^2+2\beta\cos(x)}\right) \approx$$

$\approx 2\beta\sin(x/2)^2$, as long as the upper limit of integration is fixed in $\mathbb{C}$.

Acknowledgments and a dedication

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It is dedicated to a lasting memory of Vladimir Gerdt!

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